

EVERY FILTER IS HOMEOMORPHIC TO ITS SQUARE

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ABSTRACT. We show that every filter \mathcal{F} on ω , viewed as a subspace of 2^ω , is homeomorphic to \mathcal{F}^2 . This generalizes a theorem of van Engelen, who proved that this holds for Borel filters.

1. INTRODUCTION

In [vE3], van Engelen obtained a purely topological characterization of filters, among the zero-dimensional Borel spaces.¹ In particular, he obtained the following result (see [vE3, Lemma 3.1]).

Theorem 1 (van Engelen). *If \mathcal{F} is a Borel filter then \mathcal{F} is homeomorphic to \mathcal{F}^2 .*

The main ingredients of his proof are the fact that every filter \mathcal{F} is Wadge equivalent to \mathcal{F}^2 (which is easy to see using the operation of intersection), a theorem of Steel from [St], and some of his previous work from [vE1]. It is natural to ask whether the assumption that \mathcal{F} is Borel is really necessary in Theorem 1. Our main result shows that it is not (see Theorem 6), and its proof only uses elementary methods.

2. NOTATION

Throughout this paper, Ω will denote a countably infinite set. A *filter* on Ω is a collection \mathcal{F} of subsets of Ω that satisfies the following conditions. We will write $X \subseteq^* Y$ to mean that $X \setminus Y$ is finite, and we will write $X =^* Y$ to mean that $X \subseteq^* Y$ and $Y \subseteq^* X$.

- (1) $\emptyset \notin \mathcal{F}$ and $\Omega \in \mathcal{F}$.
- (2) If $X \in \mathcal{F}$ and $X =^* Y \subseteq \Omega$ then $Y \in \mathcal{F}$.
- (3) If $X \in \mathcal{F}$ and $X \subseteq Y \subseteq \Omega$ then $Y \in \mathcal{F}$.
- (4) If $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$.

All filters are assumed to be on ω unless we explicitly say otherwise. A filter is *principal* if there exists $\Omega \subseteq \omega$ such that $\mathcal{F} = \{X \subseteq \omega : \Omega \subseteq^* X\}$. Define $\text{Fin}(\Omega) = \{X \subseteq \Omega : X \text{ is finite}\}$ and $\text{Cof}(\Omega) = \{X \subseteq \Omega : \Omega \setminus X \text{ is finite}\}$.

We will freely identify any collection \mathcal{X} consisting of subsets of Ω with the subspace of 2^Ω consisting of the characteristic functions of elements of \mathcal{X} . In particular, every filter on Ω will inherit the subspace topology from 2^Ω .

Date: May 13, 2016.

Key words and phrases. Filter, ideal, square, homeomorphism, semifilter.

The first-listed author was supported by the FWF grant M 1851-N35. The second-listed author was supported by the FWF grant I 1209-N25.

¹Actually, van Engelen stated his results for ideals. Using the homeomorphism $c : 2^\omega \rightarrow 2^\omega$ defined by $c(X)(n) = 1 - X(n)$ for $X \in 2^\omega$ and $n \in \omega$, one sees that his results also hold for filters.

Given a function f and a subset S of the domain of f , let $f[S] = \{f(X) : X \in S\}$ denote the image of S under f .

By *space* we will always mean separable metrizable topological space. A space is *crowded* if it is non-empty and it has no isolated points. Given spaces \mathcal{X} and \mathcal{Y} , we will write $\mathcal{X} \approx \mathcal{Y}$ to mean that \mathcal{X} and \mathcal{Y} are homeomorphic. We will be using freely the following well-known characterizations of \mathbb{Q} and 2^ω (see [vM, Theorem 1.9.6] and [vM, Theorem 1.5.5] respectively).

- If \mathcal{X} is a crowded countable space then $\mathcal{X} \approx \mathbb{Q}$.
- If \mathcal{X} is a crowded compact zero-dimensional space then $\mathcal{X} \approx 2^\omega$.

We will also assume that the reader is familiar with the basic theory of topologically complete spaces (see for example [vM, Section A.6]).

Given a collection \mathcal{X} consisting of subsets of ω and $\Omega \subseteq \omega$, define

$$\mathcal{X} \upharpoonright \Omega = \{X \cap \Omega : X \in \mathcal{X}\}.$$

Notice that $\mathcal{F} \upharpoonright \Omega = \{X \in \mathcal{F} : X \subseteq \Omega\}$ whenever \mathcal{F} is a filter and $\Omega \in \mathcal{F}$.

We conclude this section by remarking that many authors (including van Engelen in [vE3]) give a more general notion of filter than the one we gave above. The most general notion possible seems to be the following. Define a *prefilter* on Ω to be a collection \mathcal{F} of subsets of Ω that satisfies conditions (3) and (4). The next proposition, which can be safely assumed to be folklore, shows that our definition does not result in any substantial loss of generality.

Proposition 2. *Let \mathcal{G} be an infinite prefilter on ω . Then either $\mathcal{G} \approx 2^\omega$ or $\mathcal{G} \approx \mathcal{F}$ for some filter \mathcal{F} .*

Proof. Let $\Omega = \omega \setminus \bigcap \mathcal{G}$, and observe that Ω is infinite because \mathcal{G} is infinite. Notice that $\mathcal{G} \upharpoonright \Omega$ is a prefilter on Ω . First assume that $\emptyset \in \mathcal{G} \upharpoonright \Omega$. This means that $\bigcap \mathcal{G} = \omega \setminus \Omega \in \mathcal{G}$, hence $\mathcal{G} = \{X \subseteq \omega : \bigcap \mathcal{G} \subseteq X\} \approx 2^\omega$.

Now assume that $\emptyset \notin \mathcal{G} \upharpoonright \Omega$. We claim that $\mathcal{G} \upharpoonright \Omega$ is in fact a filter on Ω . In order to prove this claim, it only remains to show that condition (2) is satisfied. Notice that it will be enough to show that $\text{Cof}(\Omega) \subseteq \mathcal{G} \upharpoonright \Omega$. So let $F \in \text{Fin}(\Omega)$. Since $\Omega = \omega \setminus \bigcap \mathcal{G}$ and \mathcal{G} satisfies condition (4), there must be $X \in \mathcal{G}$ such that $X \subseteq \omega \setminus F$. It follows that $\omega \setminus F \in \mathcal{G}$, hence $\Omega \setminus F \in \mathcal{G} \upharpoonright \Omega$. Finally, it is straightforward to check that $\mathcal{G} \approx \mathcal{G} \upharpoonright \Omega$. \square

3. PRELIMINARY RESULTS

The following three lemmas will be needed in the proof of Theorem 6.

Lemma 3. *Assume that \mathcal{F} is a non-principal filter and $\Omega \in \mathcal{F}$. Then $\mathcal{F} \upharpoonright \Omega \approx \mathcal{F}$.*

Proof. Fix $\Omega^* \subseteq \Omega$ such that $\Omega^* \in \mathcal{F}$ and $\Omega \setminus \Omega^*$ is infinite. This is possible because \mathcal{F} is non-principal. Fix a bijection $\sigma : \omega \setminus \Omega^* \rightarrow \Omega \setminus \Omega^*$ and let $\tau : \Omega^* \rightarrow \Omega^*$ be the identity. Set $\pi = \sigma \cup \tau$ and notice that $\pi : \omega \rightarrow \Omega$ is a bijection. Therefore, the function $h : 2^\omega \rightarrow 2^\Omega$ defined by setting $h(X) = \pi[X]$ is a homeomorphism. Furthermore, using the fact that $\Omega^* \in \mathcal{F}$, it is straightforward to check that $h[\mathcal{F}] = \mathcal{F} \upharpoonright \Omega$. This shows that $\mathcal{F} \approx \mathcal{F} \upharpoonright \Omega$. \square

Lemma 4. *Assume that \mathcal{F} is a non-principal filter. Then $\mathcal{F} \times 2^\omega \approx \mathcal{F}$.*

Proof. Fix a $\Omega \in \mathcal{F} \setminus \text{Cof}(\omega)$. This is possible because \mathcal{F} is non-principal. Let $h : 2^\Omega \times 2^{\omega \setminus \Omega} \rightarrow 2^\omega$ be the function defined by setting $h(F, X) = F \cup X$. It is clear

that h is a homeomorphism. Furthermore, using the fact that $\Omega \in \mathcal{F}$, one sees that $h[\mathcal{F} \upharpoonright \Omega \times 2^{\omega \setminus \Omega}] = \mathcal{F}$. Therefore $\mathcal{F} \upharpoonright \Omega \times 2^{\omega \setminus \Omega} \approx \mathcal{F}$. An application of Lemma 3 concludes the proof. \square

Lemma 5. *Assume that \mathcal{F} is a principal filter. Then $\mathcal{F}^2 \approx \mathcal{F}$.*

Proof. It will be enough to show that $\mathcal{F} \approx \mathbb{Q}$ or $\mathcal{F} \approx \mathbb{Q} \times 2^\omega$. Fix $\Omega \subseteq \omega$ such that $\mathcal{F} = \{X \subseteq \omega : \Omega \subseteq^* X\}$. If $\Omega \in \text{Cof}(\omega)$, then $\mathcal{F} = \text{Cof}(\omega) \approx \mathbb{Q}$. So assume that $\Omega \notin \text{Cof}(\omega)$. The proof of Lemma 4 shows that $\mathcal{F} \approx \mathcal{F} \upharpoonright \Omega \times 2^{\omega \setminus \Omega}$. Since $\mathcal{F} \upharpoonright \Omega = \text{Cof}(\Omega) \approx \mathbb{Q}$, it follows that $\mathcal{F} \approx \mathbb{Q} \times 2^\omega$. \square

4. THE MAIN RESULT

We begin by introducing some useful notation. Given $S \subseteq \omega$ such that $\omega \setminus S$ is infinite, let $\phi_S : \omega \setminus S \rightarrow \omega$ denote the unique bijection such that $m < n$ implies $\phi_S(m) < \phi_S(n)$ for all $m, n \in \omega \setminus S$. Given an infinite $\Omega \subseteq \omega$, define

$$\mathcal{D}(\Omega) = \{(X, Y) \in 2^\Omega \times 2^\Omega : X \cap Y = \emptyset\}.$$

It is easy to check that $\mathcal{D}(\Omega)$ is a closed crowded subspace of $2^\Omega \times 2^\Omega$, which implies $\mathcal{D}(\Omega) \approx 2^\omega$.

Theorem 6. *If \mathcal{F} is a filter then $\mathcal{F}^2 \approx \mathcal{F}$.*

Proof. Let \mathcal{F} be a filter. If \mathcal{F} is principal, then the desired conclusion follows from Lemma 5. So assume that \mathcal{F} is non-principal, and fix $\Omega \in \mathcal{F} \setminus \text{Cof}(\omega)$.

Let $h : 2^\Omega \times 2^\Omega \times \mathcal{D}(\omega \setminus \Omega) \rightarrow 2^\Omega \times \mathcal{D}(\omega)$ be the function defined by

$$h(F, G, X, Y) = (F \cap G, \phi_{F \cap G}[(F \setminus G) \cup X], \phi_{F \cap G}[(G \setminus F) \cup Y]),$$

and observe that h is continuous.

Let $g : 2^\Omega \times \mathcal{D}(\omega) \rightarrow 2^\Omega \times 2^\Omega \times \mathcal{D}(\omega \setminus \Omega)$ be the function defined by

$$g(H, Z, W) = (H \cup (\phi_H^{-1}[Z] \cap \Omega), H \cup (\phi_H^{-1}[W] \cap \Omega), \phi_H^{-1}[Z] \cap (\omega \setminus \Omega), \phi_H^{-1}[W] \cap (\omega \setminus \Omega)),$$

and observe that g is continuous. It is straightforward to verify that g is the inverse function of h . Therefore h is a homeomorphism.

Furthermore, it is easy to realize that

$$h[\mathcal{F} \upharpoonright \Omega \times \mathcal{F} \upharpoonright \Omega \times \mathcal{D}(\omega \setminus \Omega)] \subseteq \mathcal{F} \upharpoonright \Omega \times \mathcal{D}(\omega)$$

and

$$g[\mathcal{F} \upharpoonright \Omega \times \mathcal{D}(\omega)] \subseteq \mathcal{F} \upharpoonright \Omega \times \mathcal{F} \upharpoonright \Omega \times \mathcal{D}(\omega \setminus \Omega).$$

Since $g = h^{-1}$, it follows that $h[\mathcal{F} \upharpoonright \Omega \times \mathcal{F} \upharpoonright \Omega \times \mathcal{D}(\omega \setminus \Omega)] = \mathcal{F} \upharpoonright \Omega \times \mathcal{D}(\omega)$. Therefore $\mathcal{F} \upharpoonright \Omega \times \mathcal{F} \upharpoonright \Omega \times \mathcal{D}(\omega \setminus \Omega) \approx \mathcal{F} \upharpoonright \Omega \times \mathcal{D}(\omega)$. Finally, using Lemma 3 and Lemma 4, one sees that $\mathcal{F}^2 \approx \mathcal{F}$. \square

Corollary 7. *Fix natural numbers $m, n \geq 1$. If \mathcal{F} is a filter then $\mathcal{F}^m \approx \mathcal{F}^n$.*

5. COUNTEREXAMPLES FOR SEMIFILTERS

A *semifilter* on Ω is a collection \mathcal{F} of subsets of Ω that satisfies conditions (1), (2), and (3). All semifilters are assumed to be on ω . The following proposition shows that Theorem 6 would not hold if condition (4) were dropped from the definition of filter.

Proposition 8. *There exists a semifilter \mathcal{T} such that $\mathcal{T}^2 \not\approx \mathcal{T}$.*

Proof. Fix infinite sets Ω_1 and Ω_2 such that $\Omega_1 \cup \Omega_2 = \omega$ and $\Omega_1 \cap \Omega_2 = \emptyset$. Define

$$\mathcal{T} = \{X_1 \cup X_2 : X_1 \subseteq \Omega_1, X_2 \subseteq \Omega_2, \text{Sand } (X_1 \notin \text{Fin}(\Omega_1) \text{ or } X_2 \in \text{Cof}(\Omega_2))\},$$

and observe that \mathcal{T} is a semifilter. Furthermore, it is clear that \mathcal{T} is the union of its topologically complete subspace $\{X \subseteq \omega : X \cap \Omega_1 \notin \text{Fin}(\Omega_1)\}$ and its countable subspace $\{X_1 \cup X_2 : X_1 \in \text{Fin}(\Omega_1) \text{ and } X_2 \in \text{Cof}(\Omega_2)\}$.

The following two statements are easy to verify.

- $\text{Cof}(\Omega_2)$ is a closed subspace of \mathcal{T} that is homeomorphic to \mathbb{Q} .
- $\{X \subseteq \omega : \Omega_1 \subseteq X\}$ is a closed subspace of \mathcal{T} that is homeomorphic to 2^ω .

It follows that \mathcal{T}^2 has a closed subspace homeomorphic to $\mathbb{Q} \times 2^\omega$. Since, as is not hard to check, the space $\mathbb{Q} \times 2^\omega$ cannot be written as the union of a topologically complete subspace and a countable subspace, this concludes the proof. \square

We remark that the semifilter \mathcal{T} in the above proof is actually homeomorphic to the notable space \mathbf{T} introduced by van Douwen (unpublished, see [vEvM]). See [Me, Proposition 5.4] for more details.

In fact, the main result of [Me] shows that every homogeneous zero-dimensional Borel space that is not locally compact is homeomorphic to a semifilter. Together with [vE2, Proposition 4.1], which states that $\mathcal{X}^2 \not\approx \mathcal{X}$ for almost every homogeneous zero-dimensional Borel space \mathcal{X} of low complexity, this yields many more counterexamples as in Proposition 8.

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